A Mathematical Theory of Magnetism

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X. *A Mathematical Theory of Magnetism. By* WILLIAM THOMSON, *Esq., M.A., F.R.S.E., Fellow of St. Peter's College, Cambridge, and Professor of Natural Philosophy in the University of Glasgow. Communicated by Lieut.-Colonel* SABINE, *R.A., For. Sec. R.S.*

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Introduction.

1. THE existence of magnetism is recognized by certain phenomena of force which are attributed to it as their cause. Other physical effects are found to be produced by the same agency; as in the operation of magnetism with reference to polarized light, recently discovered by Mr. FARADAY; but we must still regard magnetic force as the characteristic of magnetism, and, however interesting such other phenomena may be in themselves, however essential a knowledge of them may be for enabling us to arrive at any satisfactory ideas regarding the physical nature of magnetism, and its connection with the general properties of matter, we must still consider the investigation of the laws, according to which the development and the action of magnetic force are regulated, to be the primary object of a Mathematical Theory in this branch of Natural Philosophy.

2. Magnetic bodies, when put near one another, in general exert very sensible mutual forces; but a body which is not magnetic, can experience no force in virtue of the magnetism of bodies in its neighbourhood. It may indeed be observed that a body, M, will exert a force upon another body A; and again, on a third body B; although when A and B are both removed to a considerable distance from M, no mutual action can be discovered between themselves: but in all such cases A and B are, when in the neighbourhood of M, temporarily magnetic; and when both are under the influence of M at the same time, they are found to act upon one another with a mutual force. All these phenomena are investigated in the mathematical theory of magnetism, which therefore comprehends two distinct kinds of magnetic action:—the mutual forces exercised between bodies possessing magnetism, and the magnetization induced in other bodies through the influence of magnets. The First Part of this paper is confined to the more *descriptive* and *positive* details of the subject, with reference to the former class of phenomena. After a sufficient foundation has been laid in it, by the mathematical exposition of the distribution of magnetism in bodies, and by the determination and expression of the general laws of magnetic force, a Second Part will be devoted to the theory of magnetization by influence, or magnetic induction.

FIRST PART.—ON MAGNETS, AND THE MUTUAL FORCES BETWEEN MAGNETS.

CHAPTER I. *Preliminary Definitions and Explanations.*

3. A magnet is a substance which intrinsically possesses magnetic properties.

A piece of loadstone, a piece of magnetized steel, a galvanic circuit, are examples of the varieties of natural and artificial magnets at present known; but a piece of soft iron, or a piece of bismuth temporarily magnetized by induction, cannot, in unqualified terms, be called a magnet.

A galvanic circuit is frequently, for the sake of distinction, called an "electro-magnet;" but, according to the preceding definition of a magnet, the simple term, without qualification, may be applied to such an arrangement. On the other hand, a piece of apparatus consisting of a galvanic coil, with a soft iron core, although often called simply "an electro-magnet," is in reality a complex arrangement involving an electro-magnet (which is intrinsically magnetic as long as the electric current is sustained) and a body transiently magnetized by induction.

4. In the following analysis of magnets, the magnetism of every magnetic substance considered, will be regarded as absolutely permanent under all circumstances. This condition is not rigorously fulfilled either for magnetized steel or for loadstone, as the magnetism of any such substance is always liable to modification by induction, and may therefore be affected either by bringing another magnet into its neighbourhood, or by breaking the mass itself and separating the fragments. When, however, we consider the magnetism of any fragment taken from a steel or loadstone magnet, the hypothesis will be that it retains without any alteration the magnetic state which it actually had in its position in the body. The general theory of the distribution of magnetism founded upon conceptions of this kind, will be independent of the truth or falseness of any such hypothesis which may be made for the sake of convenience in studying the subject; but of course any actual experiments in illustration of the analysis or synthesis of a magnet would be affected by a want of *rigidity* in the magnetism of the matter operated on. For such illustrations, electro-magnets are extremely appropriate, as in them, except during the motion by which any alteration in their form or arrangement is effected, no appreciable inductive action can exist.

5. In selecting from the known phenomena of magnetism those elementary facts which are to serve for the foundation of the theory, all complex actions, depending on the irregularities of the bodies made use of, should be excluded. Thus if we were to attempt an experimental investigation of the action between two amorphous fragments of loadstone, or between two pieces of steel magnetized by ordinary processes, we should probably fail to recognize the simple laws on which the actions, resulting from such complicated circumstances, depend; and we must look for a simpler case of magnetic action before we can make an analysis which may lead to the establishment of the fundamental principles of the theory. Much complication will be avoided if we take a case in which the irregularities of one at least of the bodies do not affect the phenomena to be considered. Now the earth, as was first shown by GILBERT, is a magnet; and its dimensions are so great that there is no sensible

variation in its action on different parts of any ordinary magnet upon which we can experiment, and consequently, in the circumstances, no complicacy depending on the actual distribution of terrestrial magnetism. We may therefore, with advantage, commence by examining the action which the earth produces upon a magnet of any kind at its surface.

6. At a very early period in the history of magnetic discovery, the remarkable property of "pointing north and south" was observed to be possessed by fragments of loadstone and magnetized steel needles. To form a clear conception of this phenomenon, we must consider the total action produced by the earth upon a magnet of any kind, and endeavour to distinguish between the effects of gravitation which the earth exerts upon the body in virtue of its weight, and those which result from the magnetic agency.

7. In the first place, it is to be remarked that the magnetic agency of the earth gives rise to no resultant force of sensible magnitude, upon any magnet with reference to which we can perform experiments, as is proved by the following observed facts.

(1.) A magnet placed in any manner, and allowed to move with perfect freedom in any horizontal direction (by being floated, for example, on the surface of a liquid), experiences no action which tends to set its centre of gravity in motion, and there is therefore no horizontal force upon the body.

(2.) The magnetism of a body may be altered in any way, without affecting its weight as indicated by a balance. Hence there can be no vertical force upon it depending on its magnetism.

8. It follows that any magnetic action which the earth can exert upon a magnet must be a couple. To ascertain the manner in which this action takes place, let us conceive a magnet to be supported by its centre of gravity* and left perfectly free to turn round this point, so that without any constraint being exerted which could balance the magnetic action, the body may be in circumstances the same as if it were without weight. The magnetic action of the earth upon the magnet gives rise to the following phenomena:—

(1.) The body does not remain in equilibrium in every position in which it may be brought to rest, as it would do did it experience no action but that of gravitation.

(2.) If the body be placed in a position of equilibrium, there is a certain axis (which, for the present, we may conceive to be found by trial), such, that if the body be turned round it, through any angle, and be brought to rest, it will remain in equilibrium.

(3.) If the body be turned through 180°, about an axis perpendicular to this, it will again be in a position of equilibrium.

* The ordinary process for finding experimentally the centre of gravity of a body, fails when there is any magnetic action to interfere with the effects of gravitation. It is, however, for our present purpose, sufficient to know that the centre of gravity exists; that is, that there is a point such that the vertical line of the resultant action of gravity passes through it, in whatever position the body be held. If it were of any consequence, a process, somewhat complicated by the magnetic action, for actually determining, by experiment, the centre of gravity of a magnet might be indicated, and thus the experimental treatment of the subject in the text would be completed.

(4.) Any motion of the body whatever, which is not of either of the kinds just described, nor compounded of the two, will bring it into a position in which it will not be in equilibrium.

(5.) The directing couple experienced by the body in any position depends solely on the angle of inclination of the axis described in (1.) to the line along which it lies when the body is in equilibrium; being independent of the position of the plane of this angle, and of the position of the body with reference to that axis.

9. From these observations we draw the conclusion that a magnet always experiences a directing couple from the earth, unless a certain axis in the body is placed in a determinate position. This line in the body is called its magnetic axis*.

10. The direction towards which the magnetic axis of the body tends in virtue of the earth's action, is called "the line of dip," or "the direction of the total terrestrial magnetic force," at the locality of the observation.

11. No further explanation regarding phenomena which depend on terrestrial magnetism is required in the present chapter; but, as the facts have been stated in part, it may be right to complete the statement, as far as regards the action experienced by a magnet of any kind when held in different positions in a given locality, by mentioning the following conclusions, deduced in a very obvious manner from the general laws of magnetic action stated below, and verified fully by experiment.

If a magnet be held with its magnetic axis inclined at any angle to the line of dip, it will experience a couple, the moment of which is proportional to the sine of the angle of inclination, acting in a plane containing the magnetic axis and the line of dip. The position of equilibrium towards which this couple tends to bring the magnetic axis is stable, and if the direction of the magnetic axis be reversed, the body may be left balanced, but it will be in unstable equilibrium.

12. The directive tendency observed in magnetic bodies, being found to depend on their geographical position, and to be related in some degree to the terrestrial poles, received the name of *polarity*; probably on account of a false hypothesis of forces exercised by the pole-star† or by the earth's poles, upon certain points of the loadstone or needle, thence called the "poles of the magnet." The terms "polarity" and "poles" are still retained, but the use of them which has very generally been made, is nearly as vague as the ideas from which they had their origin. Thus when the magnet is an elongated mass, its ends are called poles if its magnetic axis be in the direction of its length; no definite points, such as those in which the surface of the body is cut by the magnetic axis, being precisely indicated by the term as it is

* Any line in the body parallel to this might, with as good reason, be called a magnetic axis, but when we conceive the magnet to be supported by its centre of gravity, the magnetic axis is naturally taken as a line through this point.

† In the poem of Guiot de Provence (quoted in Whewell's History of the Inductive Sciences, vol. ii. p. 46), a needle is described as being magnetized and placed in or on a straw (floating on water it is to be presumed)—

"Puis se torne la pointe toute
Contre l'estoile sans doute."

generally used. If, however, the body be symmetrical about its magnetic axis, and symmetrically magnetized, whether elongated in that direction or not, the poles might be definitely the *ends of the magnetic axis* (or the points in which the surface is cut by it), unless the magnet be annular and not cut by its magnetic axis (a ring electro-magnet, for instance), in which case the ordinary conception of *poles* fails. Notwithstanding this vagueness, however, the terms poles and polarity are extremely convenient, and, with the following explanations, they will frequently be made use of in this paper.

13. Let O be any point in a magnet, and let N O S be a straight line parallel to the line defined above as the magnetic axis through the centre of gravity. If the point O, however it has been chosen, be called the centre of the magnet, the line N S, terminated either at the surface, on each side, or in any arbitrary manner, is called the magnetic axis, and the ends, N, S, of the magnetic axis are called the poles of the magnet*.

14. That pole (marked N) which points, on the whole, from the north, and in northern latitudes upwards, is called the north pole, and the other (S), which points from the south, is called the south pole.

15. The *sides* of the body towards its north pole and south pole, are said to possess "northern polarity" and "southern polarity" respectively, an expression obviously founded on the idea that the surface of a magnet may in general be contemplated as a locus of poles.

16. If a magnetic body be broken up into any number of fragments, each morsel is found to be a complete magnet, presenting in itself all the phenomena of poles and polarity. This property is generally contemplated when, in modern writings on physical subjects, polarity is mentioned as a property belonging to a solid body; and a corresponding idea is involved in the term when it is applied with reference to the electric state which Mr. FARADAY discovered to be induced in non-conductors of electricity ("dielectric"), when subjected to the influence of electrified bodies†. However different are the physical circumstances of magnetic and electric polarity, it appears that the positive laws of the phenomena are the same‡, and therefore the mathematical theories are identical. Either subject might be taken as an example of a very important branch of physical mathematics, which might be called "A Mathematical Theory of Polar Forces."

17. Although we have seen that any magnet, in general, experiences from the earth an action subject to certain very simple laws, yet the actual distribution of the magnetism which it possesses may be extremely irregular. We may certainly conceive

* A definition of poles at variance with this is adopted in some special cases, especially in that of the earth considered as a great magnet, but the manner in which the term will be used in this paper will be such as to produce no confusion on this account.

† FARADAY's Experimental Researches in Electricity, Eleventh Series.

‡ See a paper "On the Elementary Laws of Statical Electricity," published in the Cambridge and Dublin Mathematical Journal (vol. i.) in December 1845.

that if the magnetized substance be a regular crystal of magnetic iron ore, the magnetism is distributed through it according to some simple law; but by taking an amorphous and heterogeneous fragment of ore presenting magnetic properties, by magnetizing in any way an irregular mass of steel, by connecting any number of morsels of magnetic matter so as to make up a complex magnet, or by bending a galvanic wire into any form, we may obtain magnets in which the magnetic property is distributed in any arbitrary manner, however irregular. Excluding for the present the last-mentioned case, let us endeavour to form a conception of the distribution of magnetism in actually magnetized matter, such as steel or loadstone, and to lay down the principles according to which it may in any instance be mathematically expressed.

18. In general we may consider a magnet as composed of matter which is magnetized throughout, since, in general, it is found that any fragment cut out of a magnetic mass is itself a magnet possessing properties entirely similar to those which have been described as possessed by any magnet whatever. It may be however that a small portion cut out of a certain position in a magnet, may present no magnetic phenomena; and if we cut equal and similar portions from different positions, we may find them to possess magnetic properties differing to any extent both in intensity, and in the directions of their magnetic axes.

19. If we find that equal and similar portions, cut in parallel directions, from any different positions in a given magnetic mass, possess equal and similar magnetic properties, the mass is said to be uniformly magnetized.

20. In general, however, the intensity of magnetization must be supposed to vary from one part to another, and the magnetic axes of the different parts to be not parallel to one another. Hence, to lay down determinately a specification of the distribution of magnetism through a magnet of any kind, we must be able to express the *intensity* and the *direction* of magnetization at each point. Before attempting to define a standard for the numerical expression of intensity in magnetization, it will be convenient to examine the elementary laws upon which the phenomena of magnetic force depend, since it is by these effects that the nature and energy of the *magnetism* to which they are due must be estimated.

CHAPTER II. *On the Laws of Magnetic Force, and on the Distribution of Magnetism in Magnetized Matter.*

21. The object of the elementary magnetic researches of COULOMB was the determination of the mutual action between two infinitely thin, uniformly and longitudinally magnetized bars. The magnets which he used were in strictness neither uniformly nor longitudinally magnetized, such a state being unattainable by any actual process of magnetization; but, as the bars were very thin cylindrical steel wires, and were symmetrically magnetized, the resultant actions were sensibly the same as if they were in reality infinitely thin, and longitudinally magnetized; and from experi-

ments which he made, it appears that the intensity of the magnetization must have been very nearly constant from the middle of each of the bars, to within a short distance from either end, where a gradual decrease of intensity is sensible*.

22. These circumstances having been attended to, COULOMB was able to deduce from his experiments the true laws of the phenomena, and arrived at the following conclusions:—

(1.) If two thin uniformly and longitudinally magnetized bars be held near one another, an action is exerted between them which consists of four distinct forces, along the four lines joining their extremities.

(2.) The forces between like ends of the two bars are repulsive†.

(3.) The forces between unlike ends are attractive.

(4.) If the bars be held so that the four distances between their extremities, two and two, are equal, the four forces between them will be equal.

(5.) If the relative positions of the bars be altered, each force will vary inversely as the square of the mutual distance of the poles between which it acts.

23. To establish a standard for estimating the *strength* of a magnet, let us conceive two infinitely thin bars to be placed so that either end of one may be a unit of distance from an end of the other. Then, if the bars be equally magnetized, each uniformly and longitudinally, to such a degree that the force between those ends shall be unity, the strength of each bar-magnet is unity‡.

24. If any number, m, of such unit bars, of equal length, be put with like ends together, so as to constitute a single complex bar, the strength of the magnet so formed is denoted by m.

If there be any number of thin bar-magnets of equal length, and each of them of such a strength that q of them, with like ends together, would constitute a unit-bar; and if p of those bars be put with like ends together, the strength of the complex magnet so formed will be $\frac{p}{q}$.

25. If a single infinitely thin bar be magnetized to such a degree that in the same positions it would produce the same effects as a complex bar of any strength m (an integer or fraction), the strength of this magnet is denoted by m.

26. If two complex bar-magnets, of the kind described above, be put near one an-

* See note on § 38, below.

† Hence we see the propriety of the terms *north* and *south* applied to the opposite polarities of a magnet, as explained above. Thus we designate the polarity, or the imaginary magnetic matter, of the northern and southern magnetic hemispheres of the earth, as northern and southern respectively; and since the poles of ordinary magnets which are repelled by the earth's northern or southern polarity must be *similar*, these also are called northern or southern, as the case may be.

‡ The Royal Society in its Instructions for making observations on Terrestrial Magnetism adopts one foot as the unit of length; and, that force which, if acting on a grain of matter, would in one second of time generate one foot per second of velocity, as the unit of force; which is consequently very nearly $\frac{1}{32 \cdot 2}$ of the weight, in any part of Great Britain or Ireland, of one grain.

other, each bar of one will act on each bar of the other with the same forces as if all the other bars were removed. Hence, if the distance between the two poles be unity, and if the strengths of the bars be respectively m and m', (whether these numbers be integral or fractional,) the force between those poles will be mm'. If, now, the relative position of the magnets be altered, so that the distance between two poles may be f, the force between them will, according to COULOMB's law, be

$$\frac{m m'}{f^2}.$$

According to the definition given above of the strength of a simple bar-magnet, it follows that the same expression gives the force between two poles of any thin, uniformly and longitudinally magnetized bars, of strengths m and m'.

27. The *magnetic moment* of an infinitely thin, uniformly and longitudinally magnetized bar, is the product of its length into its strength.

28. If any number of equally strong, uniformly and longitudinally magnetized rectangular bars of equal infinitely small sections, be put together, with like ends towards the same parts, a complex uniformly magnetized solid of any form may be produced. The *magnetic moment* of such a magnet is equal to the sum of the magnetic moments of the bars of which it is composed.

29. The magnetic moment of any continuous solid, uniformly magnetized in parallel lines, is equal to the sum of the magnetic moments of all the thin, uniformly and longitudinally magnetized bars into which it may be divided.

It follows that the magnetic moment of any part of a uniformly magnetized mass is proportional to its volume.

30. The *intensity of magnetization* of a uniformly magnetized solid is the magnetic moment of a unit of its volume.

It follows that the magnetic moment of a uniformly magnetized solid, of any form and dimensions, is equal to the product of its volume into the intensity of its magnetization.

31. If a body be magnetized in any arbitrary, regular or irregular manner, a portion may be taken in any position, so small in all its dimensions that the distribution of magnetism through it will be sensibly uniform. The quotient obtained by dividing the magnetic moment of such a portion, in any position P, by its volume, is the *intensity of magnetization* of the substance at the point P; and a line through P parallel to its lines of magnetization, is the *direction of magnetization*, at P.

CHAPTER III. *On the Imaginary Magnetic Matter by means of which the Polarity of a Magnetized Body may be represented.*

32. It will very often be convenient to refer the phenomena of magnetic force to attractions or repulsions mutually exerted between portions of an imaginary magnetic matter, which, as we shall see, may be conceived to represent the polarity of a magnet of any kind. This imaginary substance possesses none of the primary qualities

of ordinary matter, and it would be wrong to call it either a solid, or the "magnetic fluid," or "fluids"; but, without making any hypothesis whatever, we may call it "magnetic matter," on the understanding that it possesses only the property of attracting or repelling magnets, or other portions of "matter" of its own kind, according to certain determinate laws, which may be stated as follows:—

(1.) There are two kinds of imaginary magnetic matter, northern and southern, to represent respectively the northern and southern magnetic polarities of the earth, or the similar polarities of any magnet whatever.

(2.) Like portions of magnetic matter repel and unlike portions attract, mutually.

(3.) Any two small portions of magnetic matter exert a mutual force which varies inversely as the square of the distance between them.

(4.) Two units of magnetic matter, at a unit of distance from one another, exert a unit of force, mutually.

33. If quantities of magnetic matter be measured numerically in such units, and if the positive or negative sign be prefixed to denote the *species* of matter, whether *northern* (which, by convention, we may call *positive*) or *southern*, all the preceding laws are expressed in the following proposition:—

If quantities, m *and* m′, *of magnetic matter be concentrated respectively at points at a distance,* f, *from one another, they will repel with a force algebraically equal to*

$$\frac{m\,m'}{f^2}.$$

34. It appears from the explanations given above, that the circumstances of a uniformly magnetized needle may be represented if we imagine equal quantities of northern and southern magnetic matter to be concentrated at its two poles, the numerical measure of these equal quantities being the same as that of the "strength" of the magnet.

The mutual action between two needles would thus be reduced to forces of attraction and repulsion between the portions of magnetic matter by which their poles are represented.

35. Any magnetic mass whatever may, as we have seen, be regarded as composed of infinitely small bar-magnets put together in such a way as to produce the distribution of magnetism which it actually possesses; and hence, by replacing the poles of these magnets by imaginary magnetic matter, we obtain a distribution of equal quantities of northern and southern magnetic matter through the magnetized substance, by which its actual magnetic condition may be represented. The distribution of this matter becomes very much simplified from the circumstance that we have in general unlike poles of the elementary magnets in contact, by which the opposite kinds of magnetic matter are partially (or in a class of cases *wholly**) destroyed through the interior of the body. The determination of the resulting distribution of

* In all cases when the distribution is "solenoidal." See below, Chap. V. § 68. Communicated to the Royal Society, June 20, 1850.

magnetic matter, which represents in the simplest possible manner the polarity of any given magnet, is of much interest, and even importance, in the theory of magnetism, and we may therefore make this an object of investigation, before going farther.

36. Let it be required to find the distribution of imaginary magnetic matter to represent the polarity of any number of uniformly magnetized needles, $S_1 N_1, S_2 N_2, \ldots S_n N_n$ of strengths $\mu_1, \mu_2, \ldots \mu_n$ respectively, when they are placed together, end to end (not necessarily in the same straight line).

If A denote the position occupied by S_1 when the bars are in their places; if N_1 and S_2 are placed in contact at K_1; N_2 and S_3, at K_2; and so on until we have the last magnet, with its end S_n, in contact with N_{n-1}, at K_{n-1}, and its other end, N_n, free, at a point B; we shall have to imagine

μ_1 units of southern magnetic matter to be placed at A;
μ_1 units of northern, and μ_2 units of southern matter at K_1;
μ_2 units of northern, and μ_3 of southern matter at K_2;
. . . .
μ_{n-1} units of northern, and μ_n of southern matter at K_{n-1};
and lastly, μ_n units of northern matter at B.

Hence the final distribution of magnetic matter is as follows:—

	$-\mu_1$	at A
	$\mu_1-\mu_2$	K_1
	$\mu_2-\mu_3$	K_2
	. . . .	.
	. . . .	.
	$\mu_{n-1}-\mu_n$	K_{n-}
and	μ_n	B.

37. The complex magnet $AK_1K_2 \ldots K_{n-1}B$ consists of a number of parts, each of which is uniformly and longitudinally magnetized, and it will act in the same way as a simple bar of the same length, similarly magnetized; and hence the magnetic matter which represents a bar-magnet AB of this kind is concentrated in a series of points, at the ends of the whole bar, and at all the places where there is a variation in the strength* of its magnetization.

38. If the length of each part through which the strength of the magnetism is constant, be diminished without limit, and if the entire number of the parts be increased indefinitely, a straight or curved infinitely thin bar may be conceived to be produced, which shall possess a distribution of longitudinal magnetism varying continuously from one end to the other according to any arbitrary law. If the strength of the magnetism at any point P of this bar be denoted by μ, and if $[\mu]$ and (μ) denote

* This expression is equivalent to *the product of the intensity of magnetization into the section of the bar*; and by retaining it we are enabled to include cases in which the bar is not of uniform section.

the values of μ at the points A and B, the investigation of § 36, with the elementary principles and notation of the differential calculus, leads at once to the determination of the ultimate distribution of magnetic matter by which such a bar-magnet may be represented. Thus if AP be denoted by s; μ will be a function of s, which may be supposed to be known, and its differential coefficient will express the continuous distribution of magnetic matter which replaces the group of material points at K_1, K_2, &c.; so that the entire distribution of polarity in the bar and at its ends will be as follows:— in any infinitely small length, σ, of the bar, a quantity of matter equal to

$$-\frac{d\mu}{ds}\cdot\sigma,$$

and, besides, terminal accumulations, of quantities

$$-[\mu] \text{ at A}$$

and $$(\mu) \text{ at B.}$$

It follows that if through any part of the length of a bar, the strength of the magnetism is constant, there will be no magnetic matter to be distributed through this portion of the magnet; but if the strength of the magnetism varies, then, according as it diminishes or increases from the north to the south pole of any small portion, there will be a distribution of northern or southern magnetic matter to represent the polarity which results from this variation.

Corresponding inferences may be made conversely, with reference to the distribution of magnetism, when the distribution of the imaginary magnetic matter is known. Thus COULOMB found that his long thin cylindrical bar-magnets acted upon one another as if each had a symmetrical distribution of the two kinds of magnetic matter, northern within a limited space from one end, and southern within a limited space from the other, the intermediate space (constituting generally the greater part of the bar) being unoccupied; from which we infer that no variation in the magnetism was sensible through the middle part of the bar, but that, through a limited space on each side, the intensity of the magnetization must have decreased gradually towards the ends*.

39. The distribution of magnetic matter which represents the polarity of a uniformly magnetized body of any form, may be immediately determined if we imagine

* This circumstance was alluded to above, in § 21. Interesting views on the subject of the distribution of magnetism in bar-magnets are obtained by taking arbitrary examples to illustrate the investigation of the text. Thus we may either consider a uniform bar variably magnetized, or a thin bar of varying thickness, cut from a uniformly magnetized substance; and according to the arbitrary data assumed, various remarkable results may be obtained. We shall see afterwards that any such data, however arbitrary, may be actually produced in electro-magnets, and we have therefore the means of illustrating the subject experimentally, in as complete a manner as can be conceived, although from the practical *non-rigidity* of the magnetism of magnetized substances, ordinary steel or loadstone magnets would not afford such satisfactory illustrations of *arbitrary* cases as might be desired. The distribution of longitudinal magnetism in steel needles actually magnetized in different ways, and especially "magnetized to saturation," has been the object of interesting experimental and theoretical investigations by COULOMB, BIOT, GREEN and RIESS.

it divided into infinitely thin bars, in the directions of its lines of magnetization; for each of these bars will be uniformly and longitudinally magnetized, and therefore there will be no distribution of matter except at their ends. Now the bars are all terminated on each side by the surface of the body, and consequently the whole magnetic effect is represented by a certain superficial distribution of northern and southern magnetic matter. It only remains to determine the actual form of this distribution; but, for the sake of simplicity in expression, it will be convenient to state previously the following definition, borrowed from COULOMB's writings on electricity.

40. If any kind of matter be distributed over a surface, the *superficial density* at any point is the quotient obtained by dividing the quantity of matter on an infinitely small element of the surface in the neighbourhood of that point, by the area of the element.

41. To determine the superficial density at any point in the case at present under consideration, let ω be the area of the perpendicular section of an infinitely thin uniform bar, of the solid, with one end at that point. Then, if i be the intensity of magnetization of the solid, $i\omega$ will be, as may be readily shown, the "strength" of the bar-magnet. Hence at the two ends of the bar we must suppose to be placed quantities of northern and southern imaginary magnetic matter each equal to $i\omega$. In the distribution over the surface of the given magnet, these quantities of matter must be imagined to be spread over the oblique ends of the bar. Now if θ denote the inclination of the bar to a normal to the surface through one end, the area of that end will be $\frac{\omega}{\cos\theta}$, and therefore in that part of the surface we have a quantity of matter equal to $i\omega$ spread over an area $\frac{\omega}{\cos\theta}$. Hence the superficial density is

$$i\cos\theta.$$

This expression gives the superficial density at any point, P, of the surface, and its algebraic sign indicates the kind of matter, provided the angle denoted by θ be taken between the external part of the normal, and a line drawn from P in the same direction as that of the motion of a point carried from the south pole, to the north pole, of a portion close to P, of the infinitely thin bar-magnet which we have been considering.

42. Let it be required, in the last place, to determine the entire distribution of magnetic matter necessary to represent the polarity of any given magnet.

We may conceive the whole magnetized mass to be divided into infinitely small parallelepipeds by planes parallel to three planes of rectangular coordinates. Let α, β, γ denote the three edges of one of these parallelepipeds having its centre at a point P (x, y, z). Let i denote the given intensity, and l, m, n the given direction cosines of the magnetization at P. It will follow from the preceding investigation that the polarity of this infinitely small, uniformly magnetized parallelepiped, may be represented by imaginary magnetic matter distributed over its six faces in such a manner that the density will be uniform over each face, and that the quantities of matter on the six faces will be as follows:—

$-il\,.\,\beta\gamma$, and $il\,.\,\beta\gamma$; on the two faces parallel to YOZ;
$-im\,.\,\gamma\alpha$, and $im\,.\,\gamma\alpha$; on the two faces parallel to ZOX;
$-in\,.\,\alpha\beta$, and $in\,.\,\alpha\beta$; on the two faces parallel to XOY.

Now if we consider adjacent parallelepipeds of equal dimensions, touching the six faces of the one we have been considering, we should find from each of them a second distribution of magnetic matter, to be placed upon that one of those six faces which it touches. Thus if we consider the first face $\beta\gamma$, or that of which the distance from YOZ is $x-\frac{1}{2}\alpha$; we shall have a second distribution upon it derived from a parallelepiped the coordinates of the centre of which are $x-\alpha, y, z$; and the quantity of matter in this second distribution will be

$$\left\{il+\frac{d(il)}{dx}(-\alpha)\right\}\beta\gamma.$$

This, added to that which was found above, gives

$$\frac{d(il)}{dx}(-\alpha)\,.\,\beta\gamma,\text{ or }-\frac{d(il)}{dx}\,.\,\alpha\beta\gamma$$

for the total amount of matter upon this face. Again, the quantity in the second distribution on the other face, $\beta\gamma$, is equal to

$$-\left\{il+\frac{d(il)}{dx}\,.\,\alpha\right\}\beta\gamma,$$

and therefore the total amount of matter on this face will be

$$-\frac{d(il)}{dx}\,.\,\alpha\beta\gamma.$$

By determining in a similar way the final quantities of matter on the other faces of the parallelepiped, we find that the total amount of matter to be distributed over its surface is

$$-2\left\{\frac{d(il)}{dx}+\frac{d(im)}{dy}+\frac{d(in)}{dz}\right\}\alpha\beta\gamma.$$

Now as the parallelepipeds into which we imagine the whole mass divided are infinitely small, we may substitute a continuous distribution of matter through them, in place of the superficial distributions on their faces which have been determined; and in making this substitution, the quantity of matter which we must suppose to be spread through the interior of any one of them must be half the total quantity on its surface, since each of its faces is common to it and another parallelepiped. Hence the quantity of matter to be distributed through the parallelepiped $\alpha\beta\gamma$ is equal to

$$-\left\{\frac{d(il)}{dx}+\frac{d(im)}{dy}+\frac{d(in)}{dz}\right\}\alpha\beta\gamma.$$

Besides this continuous distribution through the interior of the magnet, there must be a superficial distribution to represent the neutralized polarity at its surface. If ϱ denote the density of this distribution at any point; $[l]$, $[m]$, $[n]$ the direction cosines, and $[i]$ the intensity of the magnetization of the solid close to it; and λ, μ, ν the direction cosines of a normal to the surface, we shall have, as in the case of the uniformly magnetized solid previously considered,

$$\varrho=[i]\cos\theta=[il]\,.\,\lambda+[im]\,.\,\mu+[in]\,.\,\nu \quad \ldots\ldots\ldots\ldots \quad (1).$$

If, according to the usual definition of "density," k denote the density of the magnetic matter at P, in the continuous distribution through the interior, the expression found above for the quantity of matter in the element α, β, γ, leads to the formula

$$k=-\left\{\frac{d(il)}{dx}+\frac{d(im)}{dy}+\frac{d(in)}{dz}\right\}. \quad \ldots\ldots\ldots\ldots \quad (2).$$

These two equations express respectively the superficial distribution, and the continuous distribution through the solid, of the magnetic matter which entirely represents the polarity of the given magnet. The fact that the quantity of northern matter is equal to the quantity of southern in the entire distribution, is readily verified by showing from these formulæ, as may readily be done by integration, that the total quantity of matter is algebraically equal to nothing.

43. If there be an abrupt change in the intensity or direction of the magnetization from one part of the magnetized substance to another, a slight modification in the formulæ given above will be convenient. Thus we may take a case differing very little from a given case, but which instead of presenting finite differences in the intensity or direction of magnetization, on the two sides of any surface in the substance of the magnet, has merely very sudden continuous changes in the values of those elements: we may conceive the distribution to be made more and more nearly the same as the given distribution, with its abrupt transitions, and we may determine the limit towards which the value of the expression (2) approximates, and thus, although according to the ordinary rules of the differential calculus this formula fails in the limiting case, we may still derive the true result from it. It is very easily shown in this way, that, besides the continuous distribution given by the expression (2) applied to all points of the substance for which it does not fail, there will be a superficial distribution of magnetic matter on any surface of discontinuity; and that the density of this superficial distribution will be the difference between the products of the intensity of magnetization into the cosine of the inclination of its direction to the normal, on the two sides of the surface.

44. This result, obtained by the interpretation of formula (2) in the extreme case, might have been obtained directly from the original investigation, by taking into account the abrupt variation of the magnetization at the surface of discontinuity, as we did the abrupt termination of the magnetized substance at the boundary of the magnet, and representing the un-neutralized polarity which results, by a superficial distribution of magnetic matter.

CHAPTER IV. *Determination of the Mutual Actions between any Given Portions of Magnetized Matter.*

45. The synthetical part of the theory of magnetism has for its ultimate object the determination of the total action between two magnets, when the distribution of magnetism in each is given. The principles according to which the data of such a problem may be specified have been already laid down (§§ 28–31.), and we have seen that, with sufficient data in any case, COULOMB's laws of magnetic force are sufficient to enable us to apply ordinary statical principles to the solution of the problem. Hence the elements of this part of the theory may be regarded as complete, and we may proceed to the mathematical treatment of the subject.

46. The investigations of the preceding chapter, which show us how we may conventionally represent any given magnet, in its agency upon other bodies, by an imaginary magnetic matter distributed on its surface and through its interior; enable us to reduce the problem of finding the action between any two magnets, to the known problem of determining the resultant of the attractions or repulsions exerted between the particles of two groups of matter, according to the law of force which is met with so universally in natural phenomena. The direct formulæ applicable for this object are so readily obtained by means of the elementary principles of statics, and so well known, that it is unnecessary to cite them here, and we may regard equations (1) and (2) of the preceding chapter (§ 42.) as sufficient for indicating the manner in which the details of the problem may be worked out in any particular case. The expression for the "potential," and other formulæ of importance in LAPLACE's method of treating this subject, are given below (§ 51.), as derived from the results expressed in equations (1) and (2).

47. The preceding solution of the problem, although extremely simple and often convenient, must be regarded as very artificial, since in it the resultant action is found by the composition of mutual actions between the particles of an imaginary magnetic matter, which are not the same as the real mutual actions between the different parts of the magnets themselves, although the resultant action between the entire groups of matter is necessarily the same as the real resultant action between the entire magnets. Hence it is very desirable to investigate another solution, of a less artificial form, in which the required resultant action may be obtained by compounding the real actions between the different parts into which we may conceive the magnets to be divided. The remainder of the chapter, after some preliminary explanations and definitions, will be devoted to this object.

48. The "resultant magnetic force at any point" is an expression which will very frequently be employed in what follows, and it is therefore of importance that its signification should be clearly defined. For this purpose, let us consider separately the cases of an external point in the neighbourhood of a magnet, and a point in space which is actually occupied by magnetic matter.

(1.) The resultant force at a point in space, void of magnetized matter, is the force that the north pole of a unit-bar (or a positive unit of imaginary magnetic matter), if placed at this point, would experience.

(2.) The resultant force at a point situated in space occupied by magnetized matter, is an expression the signification of which is somewhat arbitrary. If we conceive the magnetic substance to be removed from an infinitely small space round the point, the preceding definition would be applicable; since, if we imagine a very small bar-magnet to be placed in a definite position in this space, the force upon either end would be determinate. The circumstances of this case are made clear by considering the distribution of imaginary magnetic matter required to represent the given magnet, without the small portion we have conceived to be removed from its interior; which will differ from the distribution that represents the entire given magnet, in wanting the small portion of the continuous interior distribution corresponding to the removed portion, and in having instead a superficial distribution on the small internal surface bounding the hollow space. If we consider the portion removed to be infinitely small, the want of the small portion of the *solid* magnetic matter will produce no finite effect upon any point; but the superficial distribution at the boundary of the hollow space will produce a finite force upon any magnetic point within it. Hence the resultant force upon the given point round which the space was conceived to be hollowed, may be regarded as compounded of two forces, one due to the polarity of the complete magnet, and the other to the superficial polarity left free by the removal of the magnetized substance*. The former component is the force meant by the expression "the resultant force at a point within a magnetic substance," when employed in the present paper†.

49. The conventional language and ideas with reference to the imaginary magnetic

* If the portion removed be spherical and infinitely small, it may be proved that the force at any point within it, resulting from the free polarity of the solid at the surface bounding the hollow space, is in the direction of the lines of magnetization of the substance round it, and is equal to $\frac{4\pi i}{3}$. This theorem (due to POISSON) will be demonstrated at the commencement of the Theory of Magnetic Induction, because we shall have to consider the "magnetizing force" upon any small portion of an inductively magnetized substance as the actual resultant force that would exist within the hollow space that would be left if the portion considered were removed, and the magnetism of the remainder constrained to remain unaltered.

† If we imagine a magnet to be divided into two parts by any plane passing through the line of magnetization at any internal point, P, and if we imagine the two parts to be separated by an infinitely small interval and a unit north pole to be placed between them at P, the force which this pole would experience is "the resultant force at a point, P, of the magnetic substance." This is the most direct definition of the expression that could have been given, and it agrees with the definition I have actually adopted; but I have preferred the explanation and statement in the text, as being practically more simple, and more directly connected with the various investigations in which the expression will be employed.

[Note added June 15, 1850.—Some subsequent investigations on the comparison of common magnets and electro-magnets have altered my opinion, that the definition in the text is to be preferred; and I now believe the definition in the note to present the subject in the simplest possible manner, and in that which, for the applications to be made in the continuation of this Essay, is most convenient on the whole.]

matter, explained above (§§ 32–44), enable us to give the following simple statement of the definition, including both the cases which we have been considering.

The resultant magnetic force at any point, whether in the neighbourhood of a magnet or in its interior, is the force that a unit of northern magnetic matter would experience if it were placed at that point, and if all the magnetized substance were replaced by the corresponding distribution of imaginary magnetic matter.

50. The determination of the resultant force at any point is, as we shall see, much facilitated by means of a method first introduced by LAPLACE in the mathematical treatment of the theory of attraction, and developed to a very remarkable extent by GREEN in his "Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism" (Nottingham, 1828), and in his other writings on the same and on allied subjects in the Cambridge Philosophical Transactions, and in the Transactions of the Royal Society of Edinburgh. LAPLACE's fundamental theorem is so well known that it is unnecessary to demonstrate it here; but for the sake of reference, the following enunciation of it is given. The term "potential," defined in connection with it, was first introduced by GREEN in his Essay (1828). It was at a later date introduced independently by GAUSS, and is now in very general use.

Theorem (LAPLACE).—The resultant force produced by a body, or a group of attracting or repelling particles, upon a unit particle placed at any point P, is such that the difference between the values of a certain function, at any two points p and p' infinitely near P, divided by the distance pp', is equal to its component in the direction of the line joining p and p'.

Definition (GREEN).—This function, which, for a given mass, has a determinate value at any point, P, of space, is called the potential of the mass, at the point P.

It follows from the general demonstration, that, when the law of force is that of the inverse square of the distance, the potential is found by dividing the quantity of matter in any infinitely small part of the mass, by its distance from P, and adding all the quotients so obtained.

51. The same demonstration is applicable to prove, in virtue of COULOMB's fundamental laws of magnetic force, the same theorem with reference to any kind of magnet that can be conceived to be composed of uniformly magnetized bars, either finite or infinitely small, put together in any way, that is, of any magnet other than an electro-magnet; and the investigation, in the preceding chapter, of the resulting distribution of magnetic matter that may be imagined as representing in the simplest possible way the polarity of such a magnet, enables us to determine at once, from equations (1) and (2) of § 42, its potential at any point. Thus if V denote the potential at a point P, whose coordinates are ξ, η, ζ, and if dS denote an element of the surface of the magnet, situated at a point whose coordinates are $[x]$, $[y]$, $[z]$, we have, by the proposition enunciated at the end of § 49,—

$$\mathrm{V}=\iint\frac{[il]\lambda+[im]\mu+[in]\nu}{[\Delta]}d\mathrm{S}-\iiint\frac{\frac{d(il)}{dx}+\frac{d(im)}{dy}+\frac{d(in)}{dz}}{\Delta}dxdydz, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where Δ and $[\Delta]$ are respectively the distances of the points x, y, z and $[x, y, z]$ from the point P, and are given by the equations

$$\Delta^2=(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2$$

$$[\Delta]^2=(\xi-[x])^2+(\eta-[y])^2+(\zeta-[z])^2.$$

The double and triple integrals in the first and second terms of this expression are to be taken respectively over the whole surface bounding the magnet, and throughout the entire magnetized substance. Since, as is easily shown, the value of that portion of the triple integral in the second member which corresponds to an infinitely small portion of the solid containing (ξ, η, ζ), when this point is internal, is infinitely small, it follows that the magnetic force at any internal point, as defined in § 48, is derivable from a potential expressed by equation (3).

52. The expressions for the resultant force at any point, and its direction, may be immediately obtained when the potential function has been determined, by the rules of the differential calculus. Thus, if V has been determined in terms of the rectangular coordinates, ξ, η, ζ, of the point P, the three components, X, Y, Z, of the resultant force on this point will be given, in virtue of LAPLACE's fundamental theorem enunciated in § 50, by the formulæ,

$$X=-\frac{dV}{d\xi}, \quad Y=-\frac{dV}{d\eta}, \quad Z=-\frac{dV}{d\zeta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

where the negative signs are introduced, because the potential is estimated in such a way that it diminishes in the direction along which a north pole is urged. If we take the expression (3) for V, and actually differentiate with reference to ξ, η, ζ under the integral signs, we obtain expressions for X, Y, and Z which agree with the expressions that might have been obtained directly, by means of the first principles of statics (see § 46), and thus the theorem is verified. Such a verification, extended so as to be applicable to a body acting according to any law of force, constitutes virtually the ordinary demonstration of the theorem.

53. The formulæ of the preceding paragraphs are applicable for the determination of the potential, and the resultant force at any point, whether within the magnetized substance or not, according to the general definition of § 49. The case of a point in the magnetized substance, according to the conventional second definition of § 48, cannot present itself in problems with reference to the mutual action between two actual magnets. This case being therefore excluded, we may proceed to the investigations indicated in § 47.

54. In the method which is now to be followed, the magnetized substances considered must be conceived to be divided into an infinite number of infinitely small parts, and the actual magnetism of each part will be taken into account, whether in determining the potential of the magnet at a given external point, or in investigating the mutual action between two magnets. In the first place, let us determine the potential due to an infinitely small element of a magnetized substance, and for this purpose we may commence by considering an infinitely thin, uniformly magnetized

bar of finite length. If m denote the strength of the bar, and if N and S be its north and south poles respectively, its potential at any point, P, will be, according to §§ 34 and 50,

$$\frac{m}{\mathrm{NP}}-\frac{m}{\mathrm{SP}}.$$

Let Δ denote the distance of the point of bisection of the bar from P, and θ the angle between this line and the direction of the bar measured from its centre towards its north pole. Then, if a be the length of the bar, the expression for the potential becomes

$$m\left\{\frac{1}{(\Delta^2-a\Delta\cos\theta+\frac{1}{4}a^2)^{\frac{1}{2}}}-\frac{1}{(\Delta^2+a\Delta\cos\theta+\frac{1}{4}a\ )^{\frac{1}{2}}}\right\}.$$

By expanding this in ascending powers of a, and neglecting all the terms after the first, we find for the potential of an infinitely small bar magnet,

$$\frac{ma\cos\theta}{\Delta^2}.$$

If now we suppose any number of such bar-magnets to be put together so as to constitute a mass magnetized in parallel lines, infinitely small in all its dimensions, the values of θ and Δ, and consequently the value of $\frac{\cos\theta}{\Delta^2}$, will be infinitely nearly the same for all of them, and the product of this into the sum of the values of ma for all the bar-magnets will express the potential of the entire mass. Hence, if the total magnetic moment be denoted by μ, the potential will be equal to

$$\frac{\mu\cos\theta}{\Delta^2}.$$

Now if we conceive the bars to have been arranged so as to constitute a uniformly magnetized mass, occupying a volume ϕ, we should have (§ 30.) for the intensity of magnetization, $i=\frac{\mu}{\phi}$. Hence if ϕ denote the volume of an infinitely small element of uniformly magnetized matter, and i the intensity of its magnetization, the potential which it produces at any point P, at a finite distance from it, will be

$$\frac{i\phi.\cos\theta}{\Delta^2},$$

where Δ denotes the distance of P from any point, E, within the element, and θ the angle between E P and a line drawn through E, in the direction of magnetization of the element, *towards the side of it which has northern polarity*.

55. Let us now suppose the element E to be a part of a magnet of finite dimensions, of which it is required to determine the total potential at an external point, P. Let ξ, η, ζ be the coordinates of P, referred to a system of rectangular axes, and let x, y, z be those of E. We shall have

$$\Delta^2=(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2;$$

and, if l, m, n denote the direction cosines of the magnetization at E,

$$\cos\theta = l\frac{\xi-x}{\Delta}+m\frac{\eta-y}{\Delta}+n\frac{\zeta-z}{\Delta}.$$

Hence the expression for the potential of the element E becomes

$$\frac{i\phi\{l(\xi-x)+m(\eta-y)+n(\zeta-z)\}}{\{(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2\}^{\frac{3}{2}}}.$$

Now the potential of a whole is equal to the sum of the potentials of all its parts, and hence, if we take $\phi = dx\,dy\,dz$, we have, by the integral calculus, the expression,

$$V=\iiint\frac{il.(\xi-x)+im.(\eta-y)+in.(\zeta-z)}{\{(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2\}^{\frac{3}{2}}}dxdydz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

for the potential at the point P, due to the entire magnet*.

56. This expression is susceptible of a very remarkable modification, by integration by parts. Thus we may divide the second member into three terms, of which the following is one:

$$\iiint\frac{il.(\xi-x)dx}{\{(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2\}^{\frac{3}{2}}}dydz.$$

Integrating here by parts, with reference to x, we obtain

$$\left[\iint\frac{il.dy\,dz}{\Delta}\right]-\iiint\frac{\frac{d(il)}{dx}}{\Delta}dxdydz,$$

where the brackets enclosing the double integral denote that the variables in it must belong to some point of the surface. If λ, μ, ν denote the direction cosines of a normal to the surface at any point $[\xi, \eta, \zeta]$, and $d\mathrm{S}$ an element of the surface, we may take $dy\,dz=\lambda.d\mathrm{S}$, and hence the double integral is reduced to

$$\iint\frac{[il]\lambda.d\mathrm{S}}{[\Delta]};$$

and, as we readily see by tracing the limits of the first integral with reference to x, for all possible values of y and z this double integral must be extended over the entire surface of the magnet. By treating in a similar manner the other two terms of the preceding expression for V, we obtain, finally,

$$V=\iint\frac{[il]\lambda+[im]\mu+[in]\nu}{[\Delta]}d\mathrm{S}-\iiint\frac{\frac{d(il)}{dx}+\frac{d(im)}{dy}+\frac{d(in)}{dz}}{\Delta}dxdydz.$$

The second member of this equation is the expression for the potential of a certain complex distribution of matter, consisting of a superficial distribution, and a continuous internal distribution. The superficial-density of the distribution on the surface,

* From the form of definition given in the second foot-note on § 48, for the magnetic force at an internal point, it may be shown that the expression (5), as well as the expression (3), is applicable to the potential at any point, whether internal or external. The same thing may be shown by proving, as may easily be done, that the investigation of § 56 does not fail or become nugatory when (ξ, η, ζ) is included in the limits of integration.

and the density of the continuous distribution at any internal point, are expressed respectively by $[il]\lambda+[im]\mu+[in]\nu$, and $-\left\{\frac{d(il)}{dx}+\frac{d(im)}{dy}+\frac{d(in)}{dz}\right\}$. Hence we infer that the action of the complete magnet upon any external point is the same as would be produced by a certain distribution of imaginary magnetic matter, determinable by means of these expressions, when the actual distribution of magnetism in the magnet is given*. The demonstration of the same theorem, given above (§ 42), illustrates in a very interesting manner the process of integration by parts applied to a triple integral.

57. The mutual action of any two magnets, considered as the resultant of the mutual actions between the infinitely small elements into which we may conceive them to be divided, consists of a force and a couple of which the components will be expressed by means of six triple integrals. Simpler expressions for the same results may be obtained by employing a notation for subsidiary results derived from triple integration with reference to one of the bodies, in the following manner.

58. Let us in the first place determine the action exerted by a given magnet, upon an infinitely thin, uniformly and longitudinally magnetized bar, placed in a given position in its neighbourhood.

We may suppose the rectangular coordinates, ξ, η, ζ, of the north pole, and ξ', η', ζ' of the south pole of the bar to be given, and hence the components, X, Y, Z and X′, Y′, Z′, of the resultant forces, at those points, due to the other given magnet, may be regarded as known. Then, if β denote the "strength" of the bar-magnet, the components of the forces on its two poles will be respectively,

$$\beta X,\ \beta Y,\ \beta Z, \text{ on the point } (\xi, \eta, \zeta),$$

and

$$-\beta X',\ -\beta Y',\ -\beta Z', \text{ on the point } (\xi', \eta', \zeta').$$

The resultant action due to this system of forces may be determined by means of the elementary principles of statics. Thus if we conceive the forces to be transferred to the middle of the bar by the introduction of couples, the system will be reduced to a force, on this point, whose components are

$$\beta(X-X'),\quad \beta(Y-Y'),\quad \beta(Z-Z'),$$

and a couple, whose components are

$$\left\{\beta(Z+Z')\cdot\tfrac{1}{2}(\eta-\eta')-\beta(Y+Y')\cdot\tfrac{1}{2}(\zeta-\zeta')\right\},$$

$$\left\{\beta(X+X')\cdot\tfrac{1}{2}(\zeta-\zeta')-\beta(Z+Z')\cdot\tfrac{1}{2}(\xi-\xi')\right\},$$

$$\left\{\beta(Y+Y')\cdot\tfrac{1}{2}(\xi-\xi')-\beta(X+X')\cdot\tfrac{1}{2}(\eta-\eta')\right\}.$$

* This very remarkable theorem is due to Poisson, and the demonstration, as it has been just given in the text, is to be found in his first memoir on Magnetism. The demonstration which I have given in § 42 may be regarded as exhibiting, by the theory of polarity, the physical principles expressed in the analytical formulæ.

59. If l, m, n denote the direction cosines of a line drawn along the bar, from its middle towards its north pole, and if a be the length of the bar, we shall have

$$\xi-\xi'=al, \quad \eta-\eta'=am, \quad \zeta-\zeta'=an.$$

Hence, if the bar be infinitely short, and if x, y, z denote the coordinates of its middle point, we have

$$\mathrm{X}-\mathrm{X}'=\frac{d\mathrm{X}}{dx}\,.\,al+\frac{d\mathrm{X}}{dy}\,.\,am+\frac{d\mathrm{X}}{dz}\,.\,an,$$

$$\mathrm{Y}-\mathrm{Y}'=\frac{d\mathrm{Y}}{dx}\,.\,al+\frac{d\mathrm{Y}}{dy}\,.\,am+\frac{d\mathrm{Y}}{dz}\,.\,an,$$

and

$$\mathrm{Z}-\mathrm{Z}'=\frac{d\mathrm{Z}}{dx}\,.\,al+\frac{d\mathrm{Z}}{dy}\,.\,am+\frac{d\mathrm{Z}}{dz}\,.\,an.$$

Multiplying each member of these equations by β, we obtain the expressions for the components of the force in this case; and the expressions for the components of the couples are found in their simpler forms, by substituting for $\xi-\xi'$, &c. their values given above; and, on account of the infinitely small factor which each term contains, taking 2X, 2Y, and 2Z, in place of $\mathrm{X}+\mathrm{X}'$, $\mathrm{Y}+\mathrm{Y}'$, and $\mathrm{Z}+\mathrm{Z}'$.

60. Let us now suppose an infinite number of such infinitely small bar-magnets to be put together so as to constitute a mass, infinitely small in all its dimensions, uniformly magnetized in the direction (l, m, n) to such an intensity that its magnetic moment is μ. We infer, from the preceding investigation, that the total action on this body, when placed at the point x, y, z, will be composed of a force whose components are

$$\mu\left(\frac{d\mathrm{X}}{dx}l+\frac{d\mathrm{X}}{dy}m+\frac{d\mathrm{X}}{dz}n\right),$$

$$\mu\left(\frac{d\mathrm{Y}}{dx}l+\frac{d\mathrm{Y}}{dy}m+\frac{d\mathrm{Y}}{dz}n\right),$$

$$\mu\left(\frac{d\mathrm{Z}}{dx}l+\frac{d\mathrm{Z}}{dy}m+\frac{d\mathrm{Z}}{dz}n\right),$$

acting at the centre of gravity of the solid supposed homogeneous; and a couple of which the components are

$$\mu(\mathrm{Z}m-\mathrm{Y}n),$$
$$\mu(\mathrm{X}n-\mathrm{Z}l),$$
$$\mu(\mathrm{Y}l-\mathrm{X}m).$$

61. The preceding investigation enables us, by means of the integral calculus, to determine the total mutual action between any two given magnets. For, if we take X, Y, Z to denote the components of the resultant force due to one of the magnets, at any point (x, y, z) of the other, and if i denote the intensity and (l, m, n) the direction of magnetization of the substance of the second magnet at this point, we may take $\mu=i\,.\,dxdydz$ in the expressions which were obtained, and they will then express

the action which one of the magnets exerts upon an element $dxdydz$ of the other. To determine the total resultant action, we may transfer all the forces to the origin of coordinates, by introducing additional couples; and, by the usual process, we find, for the mutual action between the two magnets, a force in a line through this point, and a couple, of which the components, F, G, H, and L, M, N, are given by the equations

$$\left.\begin{aligned}
F&=\iiint\left(il\frac{dX}{dx}+im\frac{dX}{dy}+in\frac{dX}{dz}\right)dxdydz\\
G&=\iiint\left(il\frac{dY}{dx}+im\frac{dY}{dy}+in\frac{dY}{dz}\right)dxdydz\\
H&=\iiint\left(il\frac{dZ}{dx}+im\frac{dZ}{dy}+in\frac{dZ}{dz}\right)dxdydz
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\left.\begin{aligned}
L&=\iiint\left\{imZ-inY+y\left(il\frac{dZ}{dx}+im\frac{dZ}{dy}+in\frac{dZ}{dz}\right)-z\left(il\frac{dY}{dx}+im\frac{dY}{dy}+in\frac{dY}{dz}\right)\right\}dxdydz\\
M&=\iiint\left\{inX-ilZ+z\left(il\frac{dX}{dx}+im\frac{dX}{dy}+in\frac{dX}{dz}\right)-x\left(il\frac{dZ}{dx}+im\frac{dZ}{dy}+in\frac{dZ}{dz}\right)\right\}dxdydz\\
N&=\iiint\left\{ilY-imX+x\left(il\frac{dY}{dx}+im\frac{dY}{dy}+in\frac{dY}{dz}\right)-y\left(il\frac{dX}{dx}+im\frac{dX}{dy}+in\frac{dX}{dz}\right)\right\}dxdydz
\end{aligned}\right\} \quad (7)$$

62. If, in the second members of these equations, we employ for X, Y, Z respectively their values obtained, as indicated in equations (4) of § 52, by the differentiation of the expression (5) for V in § 55, we obtain expressions for F, G, H, L, M, N, which may readily be put under symmetrical forms with reference to the two magnets, exhibiting the parts of those quantities depending on the mutual action between an element of one of the magnets, and an element of the other. Again, expressions exhibiting the mutual action between any element of the imaginary magnetic matter of one magnet, and any element of the imaginary magnetic matter of the other, may be found by first modifying by integration by parts, as in § 56, from the expressions which we have actually obtained for F, G, H, L, M, N; and then substituting for X, Y, and Z their values obtained by the differentiation of the expression (3) for V.

It is unnecessary here to do more than indicate how such other formulæ may be derived from those given above; for whenever it may be required, there can be no difficulty in applying the principles which have been established in this paper to obtain any desired form of expression for the mutual action between two given magnets.

§§ 63 and 64*. *On the Expression of Mutual Action between two Magnets by means of the Differential Coefficients of a Function of their relative Position.*

63. By a simple application of the theory of the potential, it may be shown that the amount of mechanical work spent or gained in any motion of a permanent magnet, effected under the action of another permanent magnet in a fixed position, depends solely on the initial and final positions, and not at all upon the positions successively occupied by the magnet in passing from one to the other. Hence the amount of work requisite to bring a given magnet from being infinitely distant from all magnetic bodies, into a certain position in the neighbourhood of a given fixed magnet, depends solely upon the distributions of magnetism in the two, and on the relative position which they have acquired. Denoting this amount by Q, we may consider Q as a function of coordinates which fix the relative position of the two magnets; and the variation which Q experiences when this is altered in any way will be the amount of work spent or lost, as the case may be, in effecting the alteration. This enables us to express completely the mutual action between the two magnets, by means of differential coefficients of Q, in the following manner:—

If we suppose one of the magnets to remain fixed during the alterations of relative position conceived to take place, the quantity Q will be a function of the linear and angular coordinates by which the variable position of the other is expressed. Without specifying any particular system of coordinates to be adopted, we may denote by $d_\xi Q$ the augmentation of Q when the moveable magnet is pushed through an infinitely small space $d\xi$ in any given direction, and by $d_\varphi Q$ the augmentation of Q when it is turned round any given axis, through an infinitely small angle $d\varphi$. Then, if F denote the force upon the magnet in the direction of $d\xi$, and L the moment round the fixed axis of all the forces acting upon it (or the component, round the fixed axis, of the resultant couple obtained when all the forces on the different parts of the magnet are transferred to any point on this axis), we shall have

$$-\mathrm{F}d\xi=d_\xi \mathrm{Q}, \quad \text{and} \quad -\mathrm{L}d\varphi=d_\varphi \mathrm{Q},$$

since a force equal to $-\mathrm{F}$ is overcome through the space $d\xi$ in the first case, and a couple, of which the moment is equal to $-\mathrm{L}$, is overcome through an angle $d\varphi$ in the second case of motion. Hence we have

$$\mathrm{F}=-\frac{d_\xi \mathrm{Q}}{d\xi}$$

$$\mathrm{L}=-\frac{d_\varphi \mathrm{Q}}{d\varphi}.$$

64. It only remains to show how the function Q may be determined when the distributions of magnetism in the two magnets and the relative positions of the bodies are

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given. For this purpose, let us consider points P and P′, in the two magnets respectively, and let their coordinates with reference to three fixed rectangular axes be denoted by x, y, z and x', y', z'; let also the intensity of magnetization at P be denoted by i, and its direction cosines by l, m, n; and let the corresponding quantities, with reference to P′, be denoted by i', l', m', n'. Then it may be demonstrated without difficulty that

$$Q=\iiiiiint dxdydzdx'dy'dz'ii'\left\{ll'\frac{d^2\frac{1}{\Delta}}{dxdx'}+lm'\frac{d^2\frac{1}{\Delta}}{dxdy'}+ln'\frac{d^2\frac{1}{\Delta}}{dxdz'},\right.$$

$$+ml'\frac{d^2\frac{1}{\Delta}}{dydx'}+mm'\frac{d^2\frac{1}{\Delta}}{dydy'}+mn'\frac{d^2\frac{1}{\Delta}}{dydz'}$$

$$\left.+nl'\frac{d^2\frac{1}{\Delta}}{dzdx'}+nm'\frac{d^2\frac{1}{\Delta}}{dzdy'}+nn'\frac{d^2\frac{1}{\Delta}}{dzdz'}\right\},$$

where, for brevity, Δ is taken to denote $\{(x-x')^2+(y-y')^2+(z-z')^2\}^{\frac{1}{2}}$, and the differentiations upon $\frac{1}{\Delta}$ are merely indicated. Now, by any of the ordinary formulæ for the transformation of coordinates, the values of x, y, z, and x', y', z', may be expressed in terms of coordinates of the point P with reference to axes fixed in the magnet to which it belongs, of the coordinates of the point P′ with reference to axes fixed in the other, and of the coordinates adopted to express the relative position of the two magnets: and so the preceding expression for Q may be transformed into an expression involving explicitly the relative coordinates, and containing the coordinates of the points P and P′ in the two bodies only as variables in integrations, the limits of which, depending only on the forms and dimensions of the two bodies, are absolutely constant. Thus Q is obtained as a function of the relative coordinates of the bodies, and the solution of the problem is complete.

There is no difficulty in working out the result by this method, so as actually to obtain either the expressions (6) and (7) of § 61, or the expressions indicated in § 62, although the process is somewhat long.

The method just explained for expressing the mutual action between two magnets in terms of a function of their relative position, has been added to this chapter rather for the sake of completing the mathematical theory of the division of the subject to which it is devoted, than for its practical usefulness in actual problems regarding magnetic force, for which the most convenient solutions may generally be obtained by some of the more synthetical methods explained in the preceding parts of the chapter. There is however a far more important application of the principles upon which this last method is founded which remains to be made. The mechanical value of a distribution of magnetism, although it has not, I believe, been noticed in any writings hitherto published on the mathematical theory of magnetism, is a subject of investigation of great interest, and, as I hope on a later occasion to have an opportunity of showing, of much consequence, on account of its maximum and minimum problems, which lead to demonstrations of important theorems in the solutions of inverse problems regarding magnetic distribution.